# Dynamical Abelian Projection of Gluodynamics [1]

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**Abstract**

Assuming the monopole dominance, that has been proved in the lattice gluodynamics, to hold in the continuum limit, we develop an effective scalar field theory for QCD at large distances to describe confinement. The approach is based on a gauge (or projection) independent formulation of the monopole dominance and manifestly Lorentz invariant.

## 1. The monopole dominance

Numerical simulations [1] show that there exist configurations $\bar{A}_\mu$ of gauge fields which dominate in the path integral for the Wilson loop expectation value, i.e. they give a main contribution to the QCD string tension and, therefore, are the most relevant ones for the confinement in QCD. In fact, if one calculates the Wilson loop average over a subset formed by these specific configurations, the difference between the string tension extracted from such average and the full QCD string tension (extracted from the Wilson loop average over all possible gauge field configurations) appears to be about eight per cent.

When taken in a specific gauge $\chi(A) = 0$ that breaks the gauge group $G$ to its maximal Abelian subgroup $G_H$ (e.g. SU(3) to U(1)×U(1)), the dominant configurations are Dirac magnetic monopoles with respect to the unbroken Abelian gauge group [2]. For this reason, the phenomenon is called the monopole dominance. It is should be noted that not in every *Abelian projection* $\chi$, the dominant configurations are monopoles [3]. If the dominant configurations are selected as those that turn into magnetic monopoles when lifted by a gauge transformation on the surface $\chi(A) = 0$, the monopole dominance may or may not occur, depending on the choice of $\chi$. This has been indeed observed in the lattice simulations [4]. In fact, the monopole dominance has been found, up to now, only in the so called maximal Abelian projection. In the continuum limit, this gauge condition has the form

$$D_\mu(A^H)A_\mu^{off} = \partial_\mu A_\mu^{off} + ig[A_\mu^H, A_\mu^{off}] = 0 \; , \tag{1}$$

where $A_\mu = A_\mu^H + A_\mu^{off}$ and $A_\mu^{H,off}$ are Abelian (diagonal) and off-diagonal components of $A_\mu$, respectively. So, not every Abelian projection can be used to select configurations relevant for the QCD confinement.

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## 2. Monopoles and Abelian projections

In what follows, the gauge group is always assumed to be SU(2) and, hence, $G_H = U(1)$. It allows one to avoid some unnecessary technicalities. A generalization does not meet any difficulty. To describe the dominant configurations via an Abelian projection, one has to choose a gauge condition $\chi$ that breaks SU(2) to $U(1)$. Given $A_\mu$, one should find a gauge group element $\Omega_\chi = \Omega_\chi(A)$ such that the gauge transformed configuration

$$A_\mu^{\Omega_\chi} = \Omega_\chi A_\mu \Omega_\chi^\dagger + i/g \ \Omega_\chi \partial_\mu \Omega_\chi^\dagger \tag{2}$$

lies on the surface $\chi = 0$, i.e. $\Omega_\chi$ satisfies the equation

$$\chi(A^{\Omega_\chi}) = 0 \ . \tag{3}$$

Suppose one can solve (3) for a generic $A_\mu$. The abelian (Maxwell) field $C_\mu^\chi$ associated with the unbroken $U(1)$ group is extracted as follows

$$A_\mu^{\Omega_\chi} = W_\mu^\chi + \tau_3 C_\mu^\chi/2 \ , \tag{4}$$
$$C_\mu^\chi = \text{tr } \tau_3 (\Omega_\chi A_\mu \Omega_\chi^\dagger + i/g \ \Omega_\chi \partial_\mu \Omega_\chi^\dagger) \ , \tag{5}$$

where $\text{tr}\tau_3 W_\mu^\chi \equiv 0$, and $\tau_a$ are the Pauli matrices, $tr\tau_b\tau_a = 2\delta_{ab}$. Following the lattice procedure, one picks up a space point $\mathbf{x}$ (time is fixed) and surrounds it by a sphere $\Sigma_x$ centered at $\mathbf{x}$. On the sphere, one takes an infinitesimal closed contour $L(\mathbf{x}_s)$ centered at $\mathbf{x}_s \in \Sigma_x$ and calculates a flux of the magnetic field $\mathbf{B}^\chi = \text{curl } \mathbf{C}^\chi$ through $L(\mathbf{x}_s)$ in the limit when $L$ shrinks to $\mathbf{x}_s$. If $C_\mu^\chi$ is regular everywhere on $\Sigma_x$, the flux always vanishes. If the flux happens to be non-zero for some $\mathbf{x}_s \in \Sigma_x$, then there are Dirac strings passing through $\mathbf{x}_s$. Repeating this procedure for all $\mathbf{x}$, one can locate a net of Dirac strings and, hence, determine a distribution of monopoles associated with the configuration $A_\mu$ and the gauge fixing $\chi$.

The procedure should be applied to all configurations $A_\mu \in [A]$ to obtain all possible monopole configurations in the Abelian projection $\chi$. Configurations in $[A]$ that give no monopole after the projection are assumed to be irrelevant and can be thrown from the sum over configurations in the Wilson loop average.

Suppose there is a monopole at $\mathbf{x}$. On the sphere $\Sigma_x$, the first term in (5) is regular so is the magnetic field associated with it. Therefore it does not give a finite contribution to the flux through an infinitesimal surface cut out from $\Sigma_x$ by the contour $L(\mathbf{x}_s)$, whereas, according to the Stocks theorem, the second term in (5) may give a finite contribution to the flux because it can contain a total derivative of multi-valued angular functions which parametrize the group element $\Omega_\chi$. That is, the net of Dirac strings as well as position of monopoles are completely specified by all $\Omega_\chi$. Since the total magnetic flux through $\Sigma_x$ must be zero (the flux of the magnetic Coulomb field of a monopole at $\mathbf{x}$ is equal to the flux carried by the Dirac string of the monopole), we obtain for the magnetic charge at $\mathbf{x}$

$$q_\chi(\mathbf{x}) = \frac{i}{4\pi} \oint_{\Sigma_x} d\sigma_j \varepsilon_{jkn} \text{tr}(e_\chi[\partial_k e_\chi, \partial_n e_\chi]) \ , \tag{6}$$
$$e_\chi = \Omega_\chi^\dagger \tau_3 \Omega_\chi \ , \qquad \text{tr } e_\chi^2 = 2 \ , \tag{7}$$

where the radius of $\Sigma_x$ tends to zero. The integer (6) classifies maps $\Sigma_x \to SU(2)/U(1) \sim S^2$ carried out by the field $e_\chi$. It determines how many times the sphere $\Sigma_x$ is wrapped around the sphere $SU(2)/U(1) \sim S^2$.

The lattice simulations show [5] that the QCD string tension does not depend on the off-diagonal element $W_\mu^\chi$ (the Abelian dominance); they are set to be zero $W_\mu^\chi = 0$ after the projection and are not accounted for in the Wilson loop average over monopoles. Photon configurations of the Maxwell field $C_\mu^\chi$ (monopole-free configurations) are also irrelevant for the string tension (the monopole dominance [1]), i.e. only the last monopole term in (5), denoted below as $\bar{C}_\mu^\chi$, is important for the confinement. Thus, the monopole dominance implies that the dominant configurations can be parametrized by the set $[\Omega_\chi]$ of $\Omega_\chi$'s with $q_\chi \neq 0$ at least at one point $\mathbf{x}$. A parametrization of the set $[\Omega_\chi]$ is technically difficult to find. It implies solving equation (3) for a generic $A_\mu$ which seems hardly possible in the maximal Abelian projection (1) where the monopole dominance is shown to occur. An alternative approach has to be developed in the continuum theory.

It should be noted that in the continuum limit, one cannot simply put $W_\mu^\chi = 0$ and count only contributions of pure monopole configurations in the Wilson loop average because Dirac monopoles are pointlike objects and, hence, have infinite magnetic field energy. Since the string tension is not sensitive to a specific form of $W_\mu^\chi$, we choose $W_\mu^\chi = \bar{W}_\mu^\chi(\bar{C}^\chi)$ to provide a finite size to the Dirac monopoles $\bar{C}_\mu^\chi = -i/g \, \mathrm{tr}(e_\chi \Omega_\chi^\dagger \partial_\mu \Omega_\chi)$. The core functions $\bar{W}_\mu^\chi$ depend only on the monopole distributions (6). The configurations $\bar{W}_\mu^\chi + \tau_3 \bar{C}_\mu/2$ are known as the Wu-Yang monopoles with a core and their explicit construction is given in [6]. The full color *magnetic* energy $\int d^3x \mathrm{tr} F_{ij}^2/4$ is finite for these configurations. Therefore the action remains also finite at finite temperature. Note that in the lattice QCD, the lattice spacing plays the role of the monopole energy regularization.

As has been aforementioned, the Abelian projection is used only to select the dominant configurations. It does not offer any explanation why these configurations are dominant. In this regard, the monopole dominance is nothing but an "experimental" fact discovered numerically. Nevertheless, this fact can be exploited to classify the dominant configurations. To do so, we lift back the regularized monopole configurations $\bar{W}_\mu^\chi(\bar{C}^\chi) + \bar{C}_\mu^\chi$ from the surface $\chi = 0$ to the space of all configurations $[A]$ by a gauge transformation with $\Omega = \Omega_\chi^\dagger$. We denote the image of the lift $[\,{}^\chi \bar{A}] \subset [A]$. By construction, any gauge potential ${}^\chi \bar{A}_\mu$ from $[{}^\chi \bar{A}]$ becomes a monopole with a core when projected onto the gauge fixing surface $\chi = 0$, meaning that

$$ {}^\chi \bar{A}_\mu^{\Omega_\chi} = \bar{W}_\mu^\chi + \bar{C}_\mu^\chi \ . \tag{8} $$

The subset $[\,{}^\chi \bar{A}]$ is the projection-dependent (or gauge-dependent) and defined up to *regular* gauge transformations of its elements. The latter arbitrariness is associated with the freedom to redefine $\Omega_\chi$ by a shift on a regular group element, $\Omega_\chi \to \Omega_\chi \Omega_0$, such that the distribution of monopoles (6) remains untouched.

Consider a *gauge invariant* (projection independent) subset

$$ [\bar{A}] = \cup_\chi [{}^\chi \bar{A}] \subset [A] \ , \tag{9} $$

3

which is the union of the subsets $[^\chi \bar{A}]$ found in all possible Abelian projections. By definition a gauge potential $\bar{A}_\mu$ belongs to $[\bar{A}]$ if there exists an Abelian projection $\chi$ such that $\bar{A}_\mu^{\Omega_\chi}$ is a magnetic monopole with a core and $\Omega_\chi$ satisfies (3). The projection independent set $[\bar{A}]$ is formed by configurations that dominate at large distances and are responsible for generating the QCD string between two static sources. This conjecture is supported by lattice simulations. The maximal Abelian projection seems to catch a major part of $[\bar{A}]$, while some other projections do not. The latter explains the dependence of the monopole dominance on the projection recipe.

Our next problem is to find an appropriate parametrization of $[\bar{A}]$ and develop an effective field theory for dynamics in it.

## 3. Universality of the dynamical Abelian projection

In [7] the dynamical Abelian projection has been proposed. This projection does not rely on any specific gauge condition $\chi(A) = 0$ to break the gauge group to its maximal Abelian subgroup. The idea was to insert the identity

$$\sqrt{\det(-D_\mu^2)} \int \mathcal{D}\phi \exp i \int d^4 x \, \text{tr}(D_\mu \phi)^2 = 1, \tag{10}$$

where a real scalar field $\phi$ realizes the adjoint representation of the gauge group, into the integral over gauge fields. Now the gauge fields are coupled to an auxiliary scalar field in a gauge invariant way via the covariant derivative. It can be exploited to achieve an Abelian projection by imposing the following gauge condition on $\phi$: The off-diagonal components of $\phi$ are set to be zero. Positions of Dirac monopoles are determined by zeros of a gauge invariant polynom of $\phi$ that coincides with the Faddeev-Popov determinant in the unitary gauge imposed on $\phi$ [7].

To perform the dynamical Abelian projection for SU(2), one should solve (3) which assumes the form $\text{tr}(\tau_3 \Omega_\phi \phi \Omega_\phi^\dagger) = 0$, i.e., given $\phi$, we look for $\Omega_\phi \in$ SU(2) whose adjoint action on $\phi$ brings it to a diagonal form. The group element $\Omega_\phi$ is ill-defined at spacetime points where $\phi(x) = 0$. The latter condition implies three equations on four spacetime coordinates. Their solutions therefore determine world lines $x^\mu = x^\mu(\tau)$ which are shown to be world lines of magnetic monopoles [7]. To see it, we remark first that at fixed time all singularities of a generic $\Omega_\phi$ form a set of isolated points in space. Gauge potentials in the dynamical Abelian projection assume the form (2) with $\Omega_\chi = \Omega_\phi$. Let $\phi = 0$ at $\mathbf{x}$. Applying the monopole location procedure of section 2 to the Maxwell potential $C_\mu^\phi = \text{tr}(\tau_3 A_\mu^{\Omega_\phi})$, we find that the magnetic charge of the monopole at $\mathbf{x}$ is equal to $q_\phi(\mathbf{x})$ given by (6) where $e_\chi = e_\phi = \Omega_\phi^\dagger \tau_3 \Omega_\phi$.

According to (10) the scalar field fluctuates and therefore the set $[\Omega_\phi]$ of all $\Omega_\phi$'s covers *all possible* monopole configurations. Note that any distribution of monopoles $q_\phi(x)$ is determined by some $\Omega_\phi$ in (6). So, we conclude

$$[^\chi \bar{A}] \sim [\Omega_\chi] \subseteq [\Omega_\phi] \tag{11}$$

for any Abelian projection $\chi$. For a fixed distribution $q_\phi(x)$, consider a gauge potential $^\phi \bar{A}_\mu$ such that $^\phi \bar{A}_\mu^{\Omega_\phi} = \bar{W}_\mu^\phi(\bar{C}) + \tau_3 \bar{C}_\mu^\phi / 2$, where $\bar{C}_\mu^\phi = -i/g \, \text{tr}(e_\phi \Omega_\phi^\dagger \partial_\mu \Omega_\phi)$ is the Dirac

monopole potential for given $q_\phi(x)$ and $\bar{W}_\mu^\phi$ are associated core functions. Let $[^\phi \bar{A}] \subset [A]$ be a set of configurations $^\phi \bar{A}_\mu$ for all possible monopole distributions $q_\phi(x)$. Clearly, $[^\phi \bar{A}]$ is isomorphic to $[\Omega_\phi]$, and from (11) follows that $[^\phi \bar{A}]$ is larger than $[^\chi \bar{A}]$ for any $\chi$. Thus, the set $[^\phi \bar{A}]$ covers the gauge invariant set (9):

$$[\bar{A}] \subseteq [^\phi \bar{A}] \subset [A] . \tag{12}$$

Thereby, the dominant configurations $[\bar{A}]$ can be parametrized by the $\Omega_\phi$'s.

## 4. The effective scalar field theory

To develop an effective theory for dynamics in the dominant sector $[\bar{A}]$, we substitute the identity (10) in the path integral for the Wilson loop average and perform the change of variables in it

$$A_\mu = {}^\phi \bar{A}_\mu(\phi) + a_\mu , \tag{13}$$

where $^\phi \bar{A}_\mu \in [^\phi \bar{A}]$ has been described above. Since (13) is just a shift of a generic gauge field configuration on a fixed configuration parametrized by an independent integration variable $\phi$, the measure assumes the form $\mathcal{D}A_\mu \mathcal{D}\phi = \mathcal{D}a_\mu \mathcal{D}\phi$.

Assuming that the monopole dominance holds in the continuum limit, we can integrate out the small fluctuations $a_\mu$ around the dominant configurations $^\phi \bar{A}(\phi)$ by means of the Gaussian approximation. To regularize the divergency of the integral over $a_\mu$ caused by the gauge symmetry, it is suitable to fix the background gauge

$$D_\mu(^\phi \bar{A})a_\mu = 0 . \tag{14}$$

Accordingly, the associated Faddeev-Popov determinant is to be included into the path integral measure. As a result, we obtain an effective scalar field theory for the dynamics in the dominant sector $[\bar{A}]$.

Some remarks are in order. When performing an Abelian projection, some monopole distributions may multiply occur, meaning that some gauge non-equivalent potentials $A_\mu$ may lead to the same monopole distribution $q_\chi$. A monopole configuration should be taken with the weight equal to its multiplicity in the Wilson loop average over monopoles. In the effective scalar field theory constructed above, fluctuations of the monopole distribution $q_\phi(x)$ are described by angular variables $\Omega_\phi$ and zeros of $\phi$, while fluctuations of the eigenvalues of $\phi$ with a fixed distribution of zeros are associated with fluctuations of the multiplicity for a given $q_\phi$.

## 6. Conclusions

Assuming that the monopole dominance discovered in the lattice QCD survives the continuum limit, we have constructed an effective field theory for the monopole dynamics which should describe the behavior of QCD at large distances (confinement). We have also proposed a gauge (or projection) invariant formulation of the monopole dominance. The effective theory is therefore gauge invariant and respects the Lorentz symmetry.

It is worth mentioning that our approach does not suffer from the Gribov ambiguity in contrast to, for example, the maximal Abelian projection where the Gribov problem has to be resolved numerically [8] and seems hopeless to solve in the continuum limit. Note that the Gribov problem is irrelevant in the background gauge (14) because the monopole dominance justifies the perturbation expansion over $a_\mu$.

The auxiliary scalar field $\phi$ can also be viewed as a *field collective coordinate* parametrizing the dominant configurations in QCD at large distances, whereas the trick of inserting the identity (10) into the gauge field path integral is nothing but a way to obtain a right gauge and Lorentz invariant measure for the collective coordinate.

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